Exercise sheet 7: Conserved quantities, symmetries, and Einstein's equations

Please prepare your solutions, ready to present in the class on **15.06.2022** at **16:00**.

1. A vector field K^μ is a called a *Killing vector field* if it satisfies Killing's equation
$$\nabla_{(\mu}K_{\nu)} = 0 \quad \text{for all } \mu, \nu. \tag{1}$$
Suppose K^μ is a Killing vector field for a spacetime with metric $g_{\mu\nu}$.
   (a) Suppose, in some choice of coordinates, the metric is independent of a particular coordinate x^{σ^*}, that is $\partial_{\sigma^*} g_{\mu\nu} = 0$ for all μ, ν. Show that K^μ such that $K^\mu\partial_\mu = \partial_{\sigma^*}$ is a Killing vector field.
   (b) There are six linearly independent Killing vector fields for 3-dimensional euclidean spacetime. What are they in cartesian coordinates? (*Hint: use part (a), and the representation of the metric in different choices of cylindrical coordinates.*)
   (c) For a geodesic with tangent vector p^μ, show that $K^\mu p_\mu$ is conserved.
   (d) If $T_{\mu\nu}$ is the stress-energy tensor, show that $\nabla^\mu(T_{\mu\nu}K^\nu) = 0$.

2. Let K^μ be a Killing vector field for a spacetime (as defined in question 1). Show that
$$\nabla_\mu\nabla_\nu K^\rho = R^\rho{}_{\nu\mu\sigma}K^\sigma.$$
Use this to show that the Riemann scalar curvature R is conserved in the direction K^μ.

3. For a given vector field V on a spacetime, the *Lie derivative* along V is an operator $\mathcal{L}_V$ defined on tensors of arbitrary type. For functions, it acts as the directional derivative, that is $\mathcal{L}_V(f) = V(f) := V^\mu\partial_\mu f$. For vector fields W, it is defined as
$$\mathcal{L}_V(W) = [V, W].$$
   (a) Show that $\mathcal{L}_V$ satisfies the Leibniz rule $\mathcal{L}_V(fW) = f\mathcal{L}_V(W) + W\mathcal{L}_V(f)$ for all functions f and vector fields W.
   (b) How does $\mathcal{L}_V$ act on 1-forms ω? (*Hint: apply $\mathcal{L}_V$ to the scalar $\omega_\mu W^\mu$.*)
   (c) A vector field V is said to generate an isometry of a metric g if $\mathcal{L}_V(g) = 0$. Show that $\mathcal{L}_V(g) = 0$ implies that V satisfies Killing's equation (1).

4. (a) Write down Einstein's equations, and describe in words what each of the terms mean.
   (b) For a generic 4-dimensional spacetime, how many independent components do Einstein's equations represent?
   (c) Show that the vacuum Einstein's equations in 2 dimensions are satisfied by any metric.